Nonlinear Mechanical Model for the Description of Propellant Sloshing

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Shortly before resonance, the liquid with a free surface in a laterally oscillating container ceases to oscillate about its nodal diameter. The stable planar motion of the liquid shifts into an erratic fluid surface motion, i.e., an unstable motion for which the motion of the nodal diameter changes constantly. At a further small increase of the forcing frequency, a stable nonplanar motion occurs. The present paper attempts to describe the planar and "rotary" sloshing motion of the liquid by a simple analytical mechanical model that consists of a mass point constraint to a parabolic surface. It is connected to a nonlinear spring. It was found that the change of two system parameters and a third-order nonlinear spring describes the liquid motion in a cylindrical container. Test results agree very well with the analytical data.

Nomenclature

a = radius of container
f_s = equation of constraint
g = longitudinal acceleration of space vehicle, or gravity
h = liquid height
h_0 = location of nonsloshing mass
h_s = height of sloshing part of liquid
h_n = location of nth sloshing mass
k_n = spring coefficient for nth spring
m = liquid mass
m_0 = nonsloshing mass
m_s = sloshing mass
r, θ, x = cylindrical coordinates
$r_{c.g.}$ = radial displacement of c.g. of liquid
r_s = radial displacement of c.g. of sloshing part of the liquid
$x_{c.g.}$ = vertical displacement of c.g. of liquid
$\bar{x}$ = liquid amplitude measured from its undisturbed position
x_s = vertical displacement of c.g. of sloshing part of liquid; also slosh mass displacement in x direction
$\bar{x}_w$ = liquid amplitude at tank wall
$y_s(y_n)$ = slosh mass displacement in y direction
y_0 = excitation amplitude
$z_s(z_n)$ = slosh mass displacement in z direction
C_s = $\epsilon_1 \tanh[\epsilon_1(h/a)]$, parameter
c_s = damping coefficient of sphere
c_n = damping coefficient of nth slosh mass
I_0 = moment of inertia of nonsloshing mass
I_s = moment of inertia of sphere
J_1 = Bessel function of first kind and first order
D = dissipation function
F_y = liquid force
M_z = liquid moment
T = kinetic energy
V = potential energy
$\bar{\alpha}_n$ = $C_s/2a$, model parameter
α_s = $(k_s a^{2n-2}/m_s\omega_s^2)$, model parameter
ϵ_n = zeros of first derivative of Bessel function of first kind and first order $[J'(\epsilon_n) = 0]$
$\gamma_s(\gamma_n)$ = slosh damping
$\omega_s(\omega_n)$ = natural circular slosh frequency
Ω = circular forcing frequency
φ = pitch angle
Ψ = relative angle of rotation between sphere and container bottom
η_n = Ω/ω_n, ratio of forcing to natural frequency
η_s = y_s/a
ζ_s = z_s/a

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I. Introduction

ALL experimental studies concerned with liquid sloshing caused by lateral periodic excitations have revealed a peculiar type of liquid motion and liquid instability near the lowest resonant frequency of the fluid. It has been observed that this type of sloshing motion of the liquid also occurs in containers of noncircular cross section as, for instance, in rectangular tanks, indicating that the container geometry is of no decisive influence on the occurance of such a motion. The essential feature of this motion is the fact that shortly before resonance the liquid ceases to oscillate about its nodal diameter which is perpendicular to the excitation direction. First, an erratic motion of the free-fluid surface is observed, which finally, after a further increase of excitation frequency, will exhibit a fluid motion having a rotating nodal diameter. This motion invariably occurs whether or not the liquid has any initial rotation.[1] It can, however, be initiated at any earlier excitation frequency by introducing some additional rotational motion of the liquid, as could happen during the drainage of the container by the formation of a vortex. Since this phenomenon nearly can be eliminated by proper annular ring-baffles at the container wall, which may provide appropriate damping and thus smaller surface amplitudes of the liquid, the problem is recognized as being created essentially by the nonlinear effects of the fluid motion.

For a circular cylindrical container undergoing translatory excitation, Hutton[2] presented the theory about the surface motion and compared it with experimental results. No attempt, however, was made to determine the pressure distribution, the liquid force, and the moment. Some theoretical and experimental data concerning the response function of the fluid force have been given by Abramson, Chu, and Kana.[3] The analysis as well as the experiments show that there are three basic regions of liquid motion: 1) stable planar motion; 2) erratic fluid surface motion in a narrow frequency band before the resonance, i.e., unstable motion in which the motion of the nodal diameter changes constantly; and 3) stable nonplanar motion in a certain frequency band immediately above resonance.

For simplification in the treatment of the equations of motion of large liquid-propelled missiles and space vehicles, an equivalent mechanical model that should describe the motion of the liquid has to be employed. In a frequency range outside a certain domain around the resonance of the liquid, the usual sloshing motion is maintained about a stationary nodal line. The response of the propellant in this frequency region can be predicted closely by linear theory and by a mechanical model based on linearized theory.[4] This model consists essentially of a suitable mass with moment of inertia that is attached rigidly to the tank wall, and also of independently oscillating sloshing masses for each vibration mode, which are attached through springs to the container walls. Instead of a mass-spring-system one also can employ a mathematical pendulum model. In the neighborhood of the natural frequency, however, these models cannot describe accurately the observed motion of the liquid. A first attempt was made by Berlot[5] and Freed[6] with a spherical pendulum. As is known, a spherical pendulum with sinusoidally excited support will deviate from its planar motion in the plane of excitation as soon as the excitation frequency approaches the natural frequency of the system. The linearized theory of such a parametrically excited system, however, can give only a rough representation of the actual rotary sloshing motion of the liquid. It is able to predict only the general behavior of the motion.

Since the phenomenon essentially involves nonlinear effects, an extension of the theory of the spherical pendulum[7] involving nonlinear terms in the amplitude seemed logical, and it provided a fair approximation for the description of the liquid phenomenon. This nonlinear analysis exhibits the same three types of motion as the liquid system. It also established[2] the fact that the rotary motion arises as a consequence of a nonlinear coupling between liquid motion, parallel and perpendicular to the plane of excitation, and that this coupling takes place through the free-fluid surface waves. The results of these investigations exhibit clearly the liquid behavior in stable planar motion as to be that of a softening restoring characteristic, and in nonplanar motion to be that of a hardening characteristic. Experimental data reveal that the predicted nonplanar motion is in poor agreement with the experimental points.

The purpose of this paper is to derive a mechanical model that will describe the motion of the liquid and the points of instability more accurately than all previous attempts and that by proper choice of a semiempirical parameter will perform this task for different container geometries. It consists of a mass having moment of inertia, being rigidly attached to the container wall, and having independently oscillating mass points for each vibration mode rolling on a guiding surface of paraboloidal shape. Each of these mass points is coupled with a nonlinear spring capable of moving up and down the longitudinal axis of the container providing the necessary hardening effect for the rotary sloshing. The complete mechanical model is derived and compared with available experimental results.

2. Mechanical Model

The introduction of a nonlinear spring in the usual spring-mass system describing linearized sloshing cannot account for the rotary motion and a true description of the nonlinear effects of the liquid, since these types of models would neglect the important vertical displacement of the center of gravity of the liquid. Since the vertical shifting of the center of gravity of the liquid is significant for large amplitudes of the liquid surface, a pendulum performing large angular displacements would be a first approximation for the description of the liquid. It also shows a softening restoring characteristic, thus behaving like the liquid in the container, which exhibits a decrease of natural frequency with increasing wave amplitude for increasing forcing frequency below the fundamental resonance. Although the spherical pendulum yields a region of unstable motion and a domain of rotary motion, and has a behavior similar to the motion of the liquid, the predicted boundaries of instability are in rather poor agreement with experimental results. For this reason, the search for a more comprehensive mechanical model was performed. In the "design" of the mechanical model, two basic features had to be observed: 1) the mechanical model should exhibit basically the same features as the liquid; i.e., it should show the same region of softening response, unstable motion, and hardening rotary motion; and 2) it should remain simple enough not to further complicate the equations of motion of a complete space vehicle. A sliding mass point on some guiding surface with an additional nonlinear spring attached to the mass should conform to the given requirements. Since the first sloshing mode plays the predominant role in containers with circular cross section, we concentrate on the description of the liquid in that particular vibration mode.

The free-fluid surface displacement as measured from the undisturbed liquid position is, for translatory excitation in a circular cylindrical container, given by[4]

$$\bar{x}(r, \theta, t) = \frac{\Omega^2}{g/a} y_0 e^{i\Omega t} \cos\theta \times \left[\frac{r}{a} + 2\sum \frac{J_1(\epsilon_n r/a)\eta_n^2}{(\epsilon_n^2 - 1)J_1(\epsilon_n)(1-\eta_n^2)}\right] \quad (2.1)$$

where Ω is the forcing frequency, g is the longitudinal acceleration, and y_0 is the forcing amplitude. The value $\eta_n = \Omega/\omega_n$ is the ratio of forcing frequency to the nth natural frequency of the liquid, and a is the radius of the container. The expression ϵ_n represents the zeros of the first derivative of the Bessel function of first kind and first order $[J_1'(\epsilon_n) = 0;\ n = 1, 2, \ldots]$. Representing (r/a) as a Bessel-series,

$$\frac{r}{a} = 2\sum \frac{J_1(\epsilon_n r/a)}{(\epsilon_n^2 - 1)J_1(\epsilon_n)}$$

and introducing it into the infinite series yields

$$\bar{x}(r, \theta, t) = \frac{2\Omega^2 y_0 e^{i\Omega t}}{g/a} \cos\theta \sum \frac{J_1(\epsilon_n r/a)}{(\epsilon_n^2 - 1)J_1(\epsilon_n)(1-\eta_n^2)}$$

The square of the natural circular frequency is given by

$$\omega_n^2 = (g/a)\epsilon_n \tanh(\epsilon_n h/a)$$

Retaining only the predominant first mode, the free-fluid surface shape becomes

$$\bar{x} = \bar{x}_w \frac{J_1(\epsilon_1 r/a)}{J_1(\epsilon_1)} \cos\theta \quad (2.2)$$

where

$$\bar{x}_w = \frac{2\Omega^2 y_0 e^{i\Omega t}}{(g/a)(\epsilon_1^2 - 1)(1-\eta_1^2)}$$

represents the fluid amplitude at the wall of the container in the plane of excitation. From this, the radial and vertical displacement of the center of gravity of the liquid is determined and yields

$$r_{c.g.} = \frac{1}{\pi a^2 h}\int_0^a \int_0^{2\pi} \int_{-(h/2)-\bar{x}}^{h/2} r^2 \cos\theta\, dr\, d\theta\, dx = \frac{a}{h\epsilon_1^2}\bar{x}_w \quad (2.3)$$

and

$$x_{c.g.} = \frac{-1}{\pi a^2 h}\int_0^a \int_0^{2\pi} \int_{-(h/2)-\bar{x}}^{h/2} xr\, dr\, d\theta\, dx = -\frac{(\epsilon_1^2 - 1)}{4\,\epsilon_1^2 h}\bar{x}_w^2 \quad (2.4)$$

where the integral relations

$$\int_0^a r^2 J_1\left(\epsilon_1 \frac{r}{a}\right) dr = \frac{a^3 J_1(\epsilon_1)}{\epsilon_1^2}$$

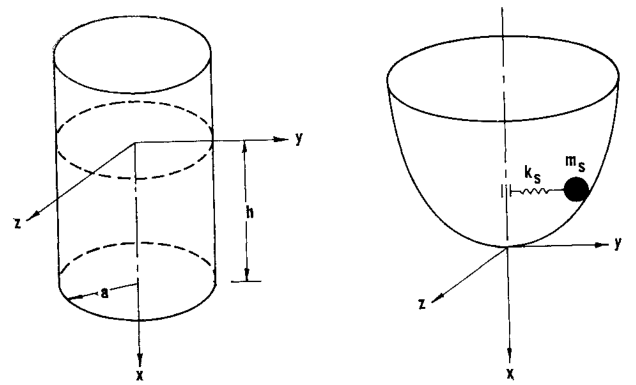

Fig. 1 Container geometry and coordinate system.

and

$$\int_0^a r\, J_1^2\left(\epsilon_1 \frac{r}{a}\right) dr = \frac{1}{2}\, a^2\, \frac{(\epsilon_1^2 - 1)}{\epsilon_1^2}\, J_1^2(\epsilon_1)$$

have been used. Both values $r_{c.g.}$ and $x_{c.g.}$ are measured from the position of the center of gravity of the undisturbed liquid. Eliminating $\bar{x}_w$ from Eqs. (2.3) and (2.4) yields the motion of the center of gravity in the form

$$x_{c.g.} = -(C/2a)r_{c.g.}^2 \tag{2.5}$$

where $C = (\epsilon_1^2 - 1)\epsilon_1^2 h/2a$.

The liquid in the lower part of an oscillating container behaves like a rigid body, and only that part of the liquid that is in the proximity of the surface oscillates in a manner that is dependent on the forcing frequency. Therefore, the liquid system can be described by a nonsloshing mass with moment of inertia, which is rigidly connected to the container and a sloshing mass point that performs oscillations while constrained by a guiding paraboloid. This system, consisting of only a mass point constrained by a paraboloid, will exhibit free rotary motion at a single frequency. No unique amplitude, however, can be determined. For this reason a nonlinear spring providing a restoring force proportional to some power of the radial displacement was introduced and provides the necessary "hardening" for the proper description of the rotary motion of the fluid. The slosh mass is attached to the spring, the other end of which is constrained to move frictionlessly up and down the center axis of the tank (Fig. 1).

2.1 Nonlinear Mechanical Slosh Model for Translatory Excitation

In the following section, we shall restrict our treatment to only the sloshing mass representing the first vibration mode of the liquid. The analytical model will be designed such that it will represent the same effect as the well-known linear mechanical models for oscillations of small amplitude response. The sloshing part of the liquid may be represented by a liquid volume of height h_s corresponding to the modal sloshing mass m_s. The ratio of the first modal sloshing mass m_s to the total liquid mass m is equal to the ratio of the height of the sloshing part of the liquid h_s to the total liquid height h, i.e.,[4]

$$\frac{h_s}{h} = \frac{m_s}{m} = \frac{2\tanh(\epsilon_1\, h/a)}{\epsilon_1\, h/a(\epsilon_1^2 - 1)} \tag{2.6}$$

The displacement of the center of gravity of the sloshing part of the liquid is, therefore, with (2.5),

$$r_s = \frac{a}{\epsilon_1^2 h_s}\,\bar{x}_w = \frac{(\epsilon_1^2 - 1)\bar{x}_w}{2\epsilon_1 \tanh(\epsilon_1\, h/a)} \tag{2.7}$$

and

$$x_s = -\frac{(\epsilon_1^2 - 1)\bar{x}_w^2}{4\epsilon_1^2 h_s} = -\frac{(\epsilon_1^2 - 1)^2\bar{x}_w^2}{8a\,\epsilon_1 \tanh(\epsilon_1\, h/a)}$$

which yield

$$x_s = -(C_s/2a)r_s^2$$

where $C_s = \epsilon_1 \tanh[\epsilon_1(h/a)]$. We proceed now to the derivation of the equations of motion of the slosh mass.

2.1.1. *Equations of motion*

The equations of motion are derived with the help of the Lagrange equation. The previously described model, subjected to a translational excitation in y direction, is employed. Viscous damping is introduced by assuming that the mass point is subjected to a damping force proportional to its velocity relative to the paraboloid.

The expression for the kinetic energy is given by

$$T = \tfrac{1}{2}m_s[(\dot{y}_s - \Omega y_0 \cos\Omega t)^2 + \dot{z}_s^2 + \dot{x}_s^2]$$

or, by the introduction of the equation of constraint,

$$f_s \equiv x_s + C_s/2a(y_s^2 + z_s^2) = 0 \tag{2.8}$$

The kinetic energy of the slosh mass yields

$$T = \tfrac{1}{2}m_s[(\dot{y}_s - \Omega y_0 \cos\Omega t)^2 + \dot{z}_s^2 + (C_s^2/a^2)(y_s\dot{y}_s + z_s\dot{z}_s)^2] \tag{2.9}$$

The potential energy is given by

$$V = -m_s g x_s + \int_0^{r_s} k_s r_s^{2n-1} dr_s$$

which yields, with the equation of constraint (2.8),

$$V = (m_s g C_s/2a)(y_s^2 + z_s^2) + (k_s/2n)(y_s^2 + z_s^2)^n \tag{2.10}$$

The first term is the gravitational potential, whereas the last term represents the energy stored in the nonlinear spring of order ($2n$-1).

The dissipation function is

$$D = \tfrac{1}{2}\bar{c}_s(\dot{x}_s^2 + \dot{y}_s^2 + \dot{z}_s^2)$$

and, with $\bar{c}_s = 2m_s\omega_s\gamma_s$ and the equation of constraint, becomes

$$D = m_s\omega_s\gamma_s[\dot{y}_s^2 + \dot{z}_s^2 + (C_s^2/a^2)(y_s\dot{y}_s + z_s\dot{z}_s)^2] \tag{2.11}$$

The introduction of these expressions into the Lagrange equation

$$(d/dt)(\partial T/\partial \dot{q}_i) - \partial T/\partial q_i + \partial D/\partial \dot{q}_i + \partial V/\partial q_i = Q_i$$

yields, with $q_1 = y_s$ and $q_2 = z_s$, the equations of motion that read with the dimensionless quantities

$$\eta_s = y_s/a \qquad \zeta_s = z_s/a \qquad \alpha_s = k_s a^{2n-2}/m_s\omega_s^2$$

$$D_1(\eta_s, \zeta_s, t) \equiv \ddot{\eta}_s + 2\omega_s\gamma_s[\dot{\eta}_s + C_s^2(\eta_s^2\dot{\eta}_s + \eta_s\zeta_s\dot{\zeta}_s)] + C_s^2[\eta_s^2\ddot{\eta}_s + \eta_s\dot{\eta}_s^2 + \eta_s\zeta_s\ddot{\zeta}_s + \eta_s\dot{\zeta}_s^2] + \omega_s^2[1 + \alpha_s(\eta_s^2 + \zeta_s^2)^{n-1}]\eta_s - \Omega^2(y_0/a)\cos\Omega t = 0 \tag{2.12}$$

and

$$D_2(\eta_s, \zeta_s, t) \equiv \ddot{\zeta}_s + 2\omega_s\gamma_s[\dot{\zeta}_s + C_s^2(\zeta_s^2\dot{\zeta}_s + \eta_s\dot{\eta}_s\zeta_s)] + C_s^2(\zeta_s^2\ddot{\zeta}_s + \zeta_s\dot{\zeta}_s^2 + \eta_s\ddot{\eta}_s\zeta_s + \dot{\eta}_s^2\zeta_s) + \omega_s^2[1 + \alpha_s(\eta_s^2 + \zeta_s^2)^{n-1}]\zeta_s = 0 \tag{2.13}$$

These equations govern the motion of the slosh mass caused by translatory excitation $y_0 \cos\Omega t$ of the system.

Although the nonlinear spring contributes to the nonlinearity of the previous equations, the principal nonlinear terms are those resulting from the vertical motion of the mass point.

It can be seen that linearization of these equations yields the results of reference[4] [Eq. (54)]. The linearized equations are

$$\ddot{\eta}_s + 2\omega_s\gamma_s\dot{\eta}_s + \omega_s^2\eta_s = \Omega^2(y_0/a)\cos\Omega t$$

and

$$\ddot{\zeta}_s + 2\omega_s\gamma_s\dot{\zeta}_s + \omega_s^2\zeta_s = 0$$

The first equation represents the motion of the slosh mass in the y direction of excitation, whereas the second equation is the free oscillation equation in the z direction (perpendicular to the excitation direction). As can be seen, the linearized equations are not coupled. Similar results can be obtained for a rotational excitation mode such as pitching or yawing.

2.1.2. Planar motion

If the liquid performs planar motion, i.e., if there is a stationary nodal line perpendicular to the direction of excitation, the coordinate ζ_s can be set equal to zero. The response is obtained by the solution of the nonlinear differential equation (2.12) with $\zeta_s = 0$. Employing the averaging method of Ritz and assuming a solution of the form

$$\bar{\eta}_s = A\cos(\Omega t + \psi) \tag{2.14}$$

where ψ is the phase angle of the motion relative to the excitation function, the Ritz conditions ($\Omega t = \tau$)

$$\int_0^{2\pi} D(\bar{\eta}_s, \tau)\cos\tau\, d\tau = 0$$

and

$$\int_0^{2\pi} D(\bar{\eta}_s, \tau)\sin\tau\, d\tau = 0$$

yield

$$\tan\psi = \frac{2\gamma_s(1 + \frac{1}{4}C_s^2A^2)}{\eta^2(1 + \frac{1}{2}C_s^2A^2) + 1 + \alpha_s[(2n)!A^{2n-2}/2^{2n-1}(n!)^2]} \tag{2.15}$$

and with $\eta^2 = \Omega^2/\omega_s^2$, the expression for the frequency response function is

$$\eta^2 = \frac{A^2(1 + \alpha_s[(2n)!A^{2n-2}/2^{2n-1}(n!)^2] \times (1 + \frac{1}{2}C_s^2A^2 - 2\gamma_s^2A^2(1 + \frac{1}{4}C_s^2A^2)^2}{A^2(1 + \frac{1}{2}C_s^2A^2)^2 - (y_0/a)^2} \pm \left(\left\{\frac{A^2(1 + \alpha_s[(2n)!A^{2n-2}/2^{2n-1}(n!')^2] \times (1 + \frac{1}{2}C_s^2A^2) - 2\gamma_s^2A^2(1 + \frac{1}{4}C_s^2A^2)^2}{A^2(1 + \frac{1}{2}C_s^2A^2)^2 - (y_0/a)^2}\right\} + \frac{A^2\{1 + \alpha_s[(2n)!A^{2n-2}/2^{2n-1}(n!)^2]\}}{(y_0/a)^2 - A^2(1 + \frac{1}{2}C_s^2A^2)^2}\right)^{1/2} \tag{2.16}$$

For zero damping ($\gamma_s = 0$), the planar undamped response is obtained with

$$\eta^2 = \frac{A\{1 + \alpha_s[A^{2n-2}(2n)!/2^{2n-1}(n!)^2]\}}{(A + \frac{1}{2}C_s^2A^3 \pm y_0/a)} \tag{2.17}$$

where the plus and minus signs in the denominator determine the response curve to the left and right of the backbone curve (in and out of phase with the excitation function). For a cubic spring, i.e., $n = 2$, the frequency response function is

$$\eta^2 = \frac{A(1 + \frac{3}{4}\alpha_sA^2)}{A + \frac{1}{2}C_s^2A^3 \pm (y_0/a)} \tag{2.18}$$

The nondimensional spring constant α_s must be determined.

A better approximation can be achieved by including an additional term in the assumed solution. This was performed for the undamped case by assuming $\bar{\eta}_s = A\cos\Omega t + B\cos 3\Omega t$. It was found by solving the occurring nonlinear algebraic equations by the Newton-Raphson method, that in the resulting solution, the maximum magnitude of B is always less than 1% of the value of A over the frequency range $0.5 \leq \eta^2 \leq 1.4$. From this one can conclude that the harmonic solution represents a very good approximation and definitely provides an acceptable degree of accuracy.

2.1.3. Nonplanar motion

For a narrow exciting frequency range that extends slightly above the linear resonance, the liquid exhibits a motion that is characterized by a steady rotation of the nodal diameter of the fluid surface. This is a stable nonplanar motion which is also termed "rotary sloshing." In this case the perpendicular coordinate ζ_s is no longer zero, and the coupled equations (2.12) and (2.13) must be solved simultaneously. With an approximate solution of the form

$$\bar{\eta}_s = A\cos\Omega t + B\sin\Omega t$$
$$\bar{\zeta}_s = D\cos\Omega t + E\sin\Omega t$$

the Ritz conditions

$$\int_0^{2\pi} D_1(\bar{\eta}_s, \bar{\zeta}_s, \tau)\begin{Bmatrix}\cos\tau\\ \sin\tau\end{Bmatrix} d\tau = 0$$

$$\int_0^{2\pi} D_2(_s\bar{\eta}, \bar{\zeta}_s, \tau)\begin{Bmatrix}\cos\tau\\ \sin\tau\end{Bmatrix} d\tau = 0$$

yield four simultaneous nonlinear algebraic equations for the unknowns A, B, D, and E.

For undamped nonplanar motion, the value $\gamma_s = 0$ and B and D vanish. The Ritz conditions are then

$$\int_0^{2\pi} D_1(\bar{\eta}_s, \bar{\zeta}_s, \tau)\cos\tau\, d\tau = 0$$

and

$$\int_0^{2\pi} D_2(\bar{\eta}_s, \bar{\zeta}_s, \tau)\sin\tau\, d\tau = 0$$

and yield the nonlinear algebraic equations

$$-A\eta^2 - \frac{C_sA\eta^2}{2}(A^2 - E^2) + \left[1 + \alpha_s\sum_{\lambda=1}^{n-1}\frac{(n-1)!\pi(2\lambda+1)!}{(\lambda!)^2(n-\lambda-1)!}A^{2\lambda}E_n^{2n-2\lambda-2} \times \frac{(2n-2\lambda-3)}{2^{2n-3}n!(n-\lambda-2)!}\right] = \frac{y_0}{a}\eta^2 \tag{2.19}$$

and

$$-E\eta^2 + \frac{C_s^2}{2}E\eta^2(A^2 - E^2) + E\left\{1 + \alpha_s\sum_{\lambda=1}^{n-1}\frac{(n-1)!\pi(2\lambda-1)!}{\lambda![(n-\lambda-1)!]^2}A^2\lambda^{2n-2\lambda-2} \times \frac{(2n-2\lambda-1)!}{2^{2n-3}n!(\lambda-1)!}\right\} = 0 \tag{2.20}$$

For $n = 2$, i.e., a spring of third order, these equations yield the expressions

$$-A\eta^2 - (C_s^2/2)A\eta^2[A^2 - E^2] + A[1 + \tfrac{3}{4}\alpha_sA^2 + \tfrac{1}{4}\alpha_sE^2] = \eta^2(y_0/a)$$

$$-E\eta^2 + (C_s^2/2)E\eta^2(A^2 - E^2) + E(1 + \tfrac{3}{4}\alpha_sE^2 + \tfrac{1}{4}\alpha_sA^2) = 0$$

These equations may be combined to yield

$$E^2 = A^2 + \frac{\eta^2(y_0/a)}{A(C_s^2\eta^2 - \frac{1}{2}\alpha_s)} \tag{2.21}$$

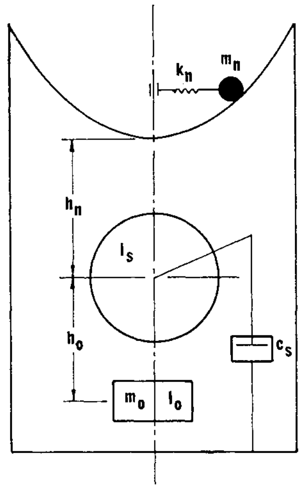

Fig. 2 Mechanical model for total liquid system

and

$$C_s^2(2A + y_0/a)\eta^4 - \alpha_s[A + \tfrac{3}{2}(y_0/a) + (2C_s^2/\alpha_s)A(1 + \alpha_s A^2)]\eta^2 + \alpha_s A(1 + \alpha_s A^2) = 0 \qquad (2.22)$$

The last equation will provide the amplitude-frequency relation in the y direction, whereas Eq. (2.21) yields, with the obtained $A = A(\eta)$, the response amplitude $E = E(\eta)$ in the z direction.

2.1.4. Derivation of the nonlinear mechanical model of the total liquid system

The analytical mechanical analogy is designed in such a fashion that it describes the observed nonlinear phenomena and presents the results of the linear model in a limit consideration for small amplitudes. The liquid in the lower part of the container follows the motion like a rigid body, and is chosen to have a mass m_0 and a moment of inertia I_0. The sloshing masses are denoted by m_n and the spring "stiffnesses" by k_n (we restrict ourselves to third-order springs). The nonsloshing mass m_0 is connected rigidly at a height h_0 below the center of gravity of the quiescent liquid. To make the mechanical model equivalent to the fluid system, the sum of the model masses must be equal to the total liquid mass. It is therefore

$$m = m_0 + \sum_{n=1}^{\infty} m_n \qquad (2.23)$$

For pitching or yawing excitation about the origin, not all of the fluid participates in the motion, but a part remains completely at rest. For this reason a frictionlessly mounted, massless sphere with a moment of inertia I_s has been introduced at the center of gravity of the quiescent liquid.

Assuming that the sloshing masses are subjected to a damping force proportional to their velocity relative to the paraboloid, the dissipation function for the nth sloshing mass point is

$$D_n(\dot{x}_n, \dot{y}_n, \dot{z}_n) = \tfrac{1}{2}\, c_n(\dot{x}_n^2 + \dot{y}_n^2 + \dot{z}_n^2)$$

where $c_n = 2m_n\omega_n\gamma_n$. In addition, a damper is introduced with damping coefficients c_{s1} and c_{s2} between the sphere and the container bottom. The reason for this is the fact that for rotational excitation of a viscous liquid, more fluid participates in the motion than for frictionless liquid.

The equations of motion of the mechanical model are now derived with the help of the Lagrange equations. For this reason one determines the kinetic and potential energy as well as the dissipation function of the system (see Fig. 2). With x_n, y_n, and z_n as the displacement of the nth sloshing mass m_n with respect to the container, with $y(t)$ the tank displacement in y direction, with φ the rotation about the z axis, and with Ψ_1 and Ψ_2 the rotation angle in perpendicular directions of the sphere with respect to the container bottom, the kinetic energy with the equation of constraint $f \equiv x_n + \bar{\alpha}_n(y_n^2 + z_n^2) = 0$ is given by

$$T = \frac{m_0}{2}(\dot{y} - h_0\dot{\varphi})^2 + \frac{1}{2}I_0\dot{\varphi}^2 + \frac{1}{2}I_s[(\dot{\varphi} + \dot{\Psi}_1)^2 + \dot{\Psi}_2^2] + \frac{1}{2}\sum_{n=1}^{\infty} m_n[(\dot{y}_n + \dot{y} + [h_n + \bar{\alpha}_n^2(y_n^2 + z_n^2)]\dot{\varphi}^2) + \dot{z}_n^2 + [2\bar{\alpha}_n(\dot{y}_n y_n + \dot{z}_n z_n) - y_n\dot{\varphi}]^2] \qquad (2.24)$$

The dissipation function is with $c_{s1} = c_{s2}$

$$D = \frac{1}{2}\sum_{n=1}^{\infty} c_n\{\dot{y}_n^2 + \dot{z}_n^2 + [2\bar{\alpha}_n(\dot{y}_n\, y_n + \dot{z}_n\, z_n)]^2\} + \frac{c_s}{2}(\dot{\psi}_1^2 + \dot{\psi}_2^2) \qquad (2.25)$$

and the potential energy is

$$V = \frac{m_0}{2} g\, h_0\varphi^2 - \frac{g}{2}\varphi^2\sum_{n=1}^{\infty} m_n[h_n + \bar{\alpha}_n(y_n^2 + z_n^2)] - g\varphi\sum_{n=1}^{\infty} m_n\, y_n + \frac{1}{4}\sum_{n=1}^{\infty} k_n(y_n^2 + z_n^2)^2 + \sum_{n=1}^{\infty} m_n g\bar{\alpha}_n(y_n^2 + z_n^2) \qquad (2.26)$$

The equations of motion are derived from the Lagrange equations

$$(d/dt)(\partial T/\partial\dot{q}_v) + (\partial D/\partial\dot{q}_v) - (\partial T/\partial q_v) + (\partial V/\partial q_v) = Q_v \qquad (2.27)$$

where the generalized coordinates q_v are y, φ, ψ_1, ψ_2, y_n, and z_n; and $Q_y = -F_y$, $Q_\varphi = -M_z$, $Q_{\Psi_1} = Q_{\Psi_1} = 0$, and $Q_{y_n} =$

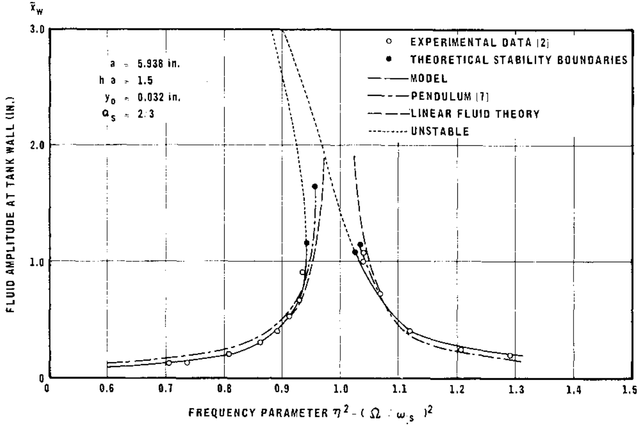

Fig. 3 Planar liquid motion; comparison of theoretical and experimental fluid amplitudes.

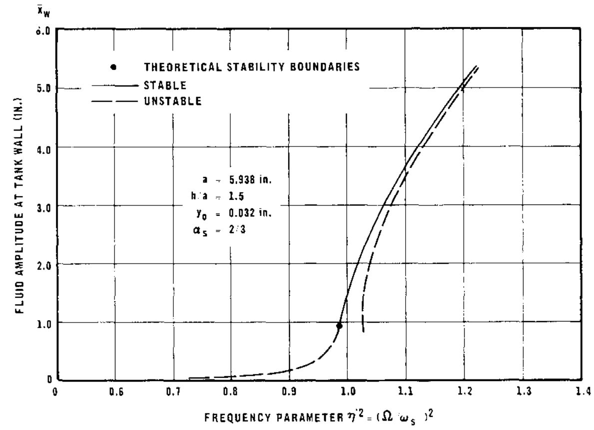

Fig. 4 Nonplanar response in direction of excitation.

$Q_{z_n} = 0$ are the generalized forces. The equations of motion are then

$$m_n(\ddot{y} - h_0\ddot{\varphi}) + \sum_{n=1}^{\infty} m_n\{\ddot{y}_n + \ddot{y} + [h_n + \bar{\alpha}_n(y_n^2 + z_n^2)]\ddot{\varphi} + 2\bar{\alpha}_n\dot{\varphi}(\dot{y}_n y_n + \dot{z}_n z_n)\} = -F_y \quad (2.28)$$

$$(m_0h_0^2 + I_0)\ddot{\varphi} - m_0h_0\ddot{y} + I_s(\ddot{\varphi} + \ddot{\Psi}_1) + \sum_{n=1}^{\infty} m_n\ddot{y}[h_n + \bar{\alpha}_n(y_n^2 + z_n^2)] + \sum_{n=1}^{\infty} m_n\ddot{y}_n[h_n + \bar{\alpha}_n(z_n^2 + y_n^2)] - 2\sum_{n=1}^{\infty} m_n\ddot{z}_n\bar{\alpha}_n y_n z_n + \ddot{\varphi}\sum_{n=1}^{\infty} m_n[2h_n\bar{\alpha}_n(y_n^2 + z_n^2) + \bar{\alpha}_n^2(y_n^2 + z_n^2)^2 + h_n^2 + y_n^2] + 2\sum_{n=1}^{\infty} m_n\bar{\alpha}_n[2(y_n^2 + z_n^2)\bar{\alpha}_n\dot{\varphi} + \dot{y}]\cdot(\dot{y}_n y_n + \dot{z}_n z_n) - 2\sum_{n=1}^{\infty} m_n y_n\bar{\alpha}_n(\dot{y}_n^2 + \dot{z}_n^2) + 2\sum_{n=1}^{\infty} m_n y_n\dot{y}_n\dot{\varphi} + m_0 g h_0\varphi - g\varphi\sum_{n=1}^{\infty} m_n[h_n + \bar{\alpha}_n(y_n^2 + z_n^2)] - g\sum_{n=1}^{\infty} m_n y_n = -M_z \quad (2.29)$$

$$I_s(\ddot{\Psi}_1 + \ddot{\varphi}) + C_s\dot{\Psi}_1 = 0 \quad (2.30)$$

$$I_s\ddot{\Psi}_2 + c_s\dot{\Psi}_2 = 0 \quad (2.31)$$

$$\ddot{y} + \ddot{y}_n + h_n\ddot{\varphi} + 4\bar{\alpha}_n^2 y_n(\dot{y}_n^2 + \dot{z}_n^2 + \ddot{y}_n y_n + \ddot{z}_n z_n) + (c_n/m_n)[\dot{y}_n + 4\bar{\alpha}_n^2 y_n(y_n\dot{y}_n + z_n\dot{z}_n)] + (k_n/m_n)y_n(y_n^2 + z_n^2) - g\varphi + 2\bar{\alpha}_n g y_n = 0 \qquad n = (1, 2, \ldots) \quad (2.32)$$

$$\ddot{z}_n + 4\bar{\alpha}_n^2 z_n(\dot{y}_n^2 + \dot{z}_n^2 + \ddot{y}_n y_n + \ddot{z}_n z_n) + (c_n/m_n)\dot{z}_n + (4c_n/m_n)\bar{\alpha}_n^2 z_n(y_n\dot{y}_n + z_n\dot{z}_n) + (k_n/m_n)z_n(y_n^2 + z_n^2) + 2\bar{\alpha}_n g z_n = 0 \qquad (n = 1, 2, \ldots) \quad (2.33)$$

The first equation is the force equation, and it was obtained with the generalized coordinate y. The second equation, as obtained with the generalized coordinate φ, is the moment equation, whereas the third and fourth equations represent the equation of motion of the sphere and were obtained with the generalized coordinates ψ_1 and ψ_2, respectively. The last two equations are the sloshing equations in y and z direction and are obtained by using the generalized coordinates y_n and z_n, respectively, in the Lagrange equation.

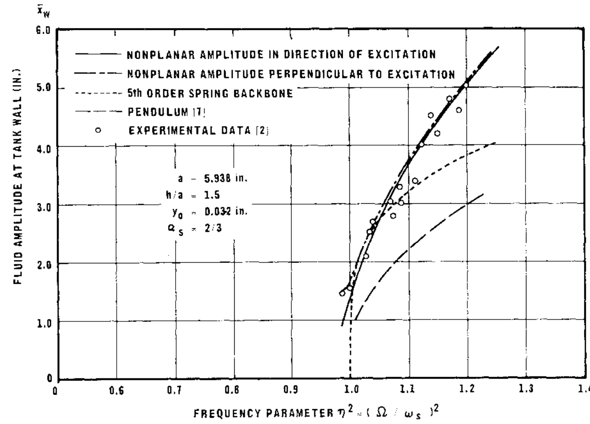

Fig. 6 Comparison of theoretical and experimental fluid amplitudes.

Linearization of this system yields the equations of motion of the linear model which agrees with the result of the linear model equations [(51–54) of Ref. 4].

In the linearized moment equation, the expression

$$m_0h_0 = \sum_{n=1}^{\infty} m_n h_n$$

expresses that the mass center of the liquid shifts only horizontally. In the linearized form there is no coupling between these motions.

3. Cylindrical Container

As mentioned previously the nondimensional spring constant α_s has to be known. Comparison of the model with Hutton's[2] test results reveals that $\frac{1}{2} \leq \alpha_s \leq \frac{2}{3}$ for a cubic spring. A cubic spring seems to describe both planar and nonplanar motions best. The response curves for planar motion are shown in Fig. 3 using $\alpha_s = \frac{2}{3}$. The dashed line represents the response function of the linearized fluid theory, whereas the results of the pendulum model are indicated also as a dash-dotted line. They exhibit a noticeable deviation. Figures 4 and 5 show the rotary response; the former exhibits the fluid amplitude in the direction of excitation, and the latter presents it in perpendicular direction. In Fig. 6 the nonplanar response is shown and exhibits good agreement with the experimental results. The backbone curve for the rotary response with a fifth-order spring is shown in the same figure for comparison. It indicates again that better agreement can be obtained with the third-order spring. The dashed line, which represents the result of the pendulum, clearly exhibits too much hardening.

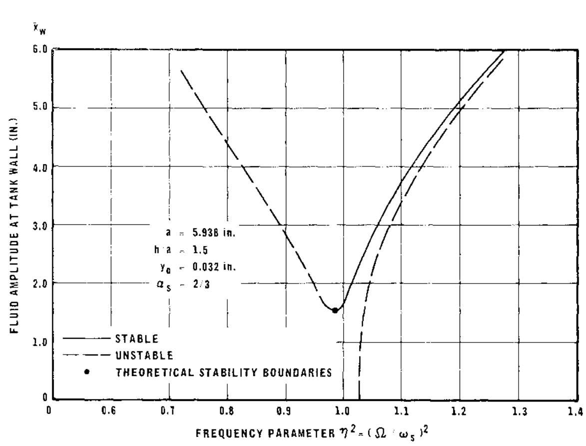

Fig. 5 Nonplanar response perpendicular to excitation direction.

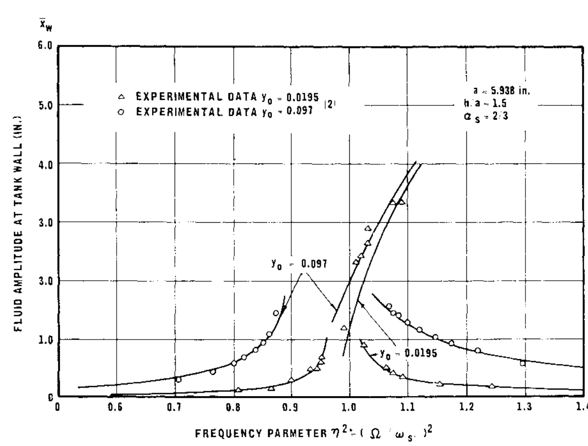

Fig. 7 Comparison of theoretical and experimental fluid amplitudes.

It may be mentioned that the stability points have been obtained by a method by Miles[1] and Hutton[2] that was readily adapted to the model equations.[8] Figure 7 exhibits the results of the model for various excitation amplitudes and compares it with available experimental results.[2] For a larger excitation amplitude, a value of $\alpha_s = \frac{1}{2}$ may yield better comparison.

4. Conclusion

The nonlinear mechanical model as derived in Sec. 2 yields, with a third-order spring for $\alpha_s = \frac{1}{2} - \frac{2}{3}$ and $C_s \approx 2$ for a cylindrical container, relatively good results for the description of the nonlinear liquid motion. It may be noted, however, that the results for α_s and C_s are based on the model tank of Ref. 2 and one liquid height, and that for other tank sizes these values may be different. It is believed furthermore that the same model can be employed for different container forms, such as rectangular or spherical tanks. The parameter α_s and C_s then can be determined from the test results and the evaluation of the analytical mechanical model.

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A Computational Method for the Optimization of Multistage Ballistic Systems

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This paper is concerned with the optimization of a multistage ballistic missile system operating in a drag environment. Some difficulties arise because of the physical discontinuities implicit in the description of such systems. The purpose of this paper is to present an integrated approach to the solution of such problems, combining both the theoretical and computational aspects necessary to handle such discontinuities. A Denbow transformation is used to remove the staging discontinuities; the resulting multipoint boundary value problem is solved numerically by the generalized Newton-Raphson method. A computer program, which embodies this approach, has been written to optimize a two-stage vehicle in a drag environment using the two-dimensional equations of motion. The angle of attack, subject to an inequality constraint, is used as a control variable. Several solutions are discussed. In the drag-free cases, the method exhibits the traditional quadratic convergence. Some difficulties have been experienced in drag cases, but a convergence parameter similar to the Gradient-Method step size has been used with some success.

1. Introduction

THE mathematical theory used for the study of optimization problems is the calculus of variations. The results most useful for attacking complicated optimization problems are the first necessary conditions on the solution, as exemplified by either Bliss' Multiplier Rule for the Problem of Bolza,[1] or Pontryagin's Maximum Principle.[2] If the equations describing the system to be optimized may be written in such a form as to satisfy the hypotheses, those conditions effectively transform a problem of optimization into the problem of finding the solutions of a two-point boundary value problem of a system of ordinary, usually nonlinear, differential equal tions with algebraic side constraints. Many computationa-methods have been developed to handle such problems. Results found by this method will be optimal in the sense that they lead to the vanishing of the first variation of the selected pay-off function. The solution point will thus be a stationary point. This property is inherent in the solution of the equations derived from the multiplier rule. The method will not necessarily yield results that satisfy the necessary conditions of Weierstrass and Clebsch or the fourth necessary condition as found in Bliss.[1] Neither will they necessarily satisfy the sufficient conditions. However, it is usually apparent from the results that a local improvement has been achieved, even if no tests are made on the second variation. Thus, the solution may not be optimal, but it is at least improved.

The optimization of multistage ballistic systems, after appropriate manipulation of the descriptive equations, can be handled by such techniques. However, some difficulties

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